Kalman Filter Properties

Sensor Fusion

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The Kalman Filter

The Kalman filter is the exact solution to the Bayesian filtering recursion for linear Gaussian model

$$\begin{aligned} x_{k+1} &= F_k x_k + G_k v_k, & \mathrm{Cov}(v_k) &= Q_k \\ y_k &= H_k x_k + e_k, & \mathrm{Cov}(e_k) &= R_k, \end{aligned}$$

assuming $\mathrm{E}(v_k) = 0$, $\mathrm{E}(e_k) = 0$, and mutual independence.

Kalman Filter Algorithm

Time update:

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k}$$

$$P_{k+1|k} = F_k P_{k|k} F_k^T + G_k Q_k G_k^T$$

Meas. update:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - \hat{y}_k)$$

$$P_{k|k} = P_{k|k-1} - K_k P_{k|k-1}$$

$$\hat{y}_k = H_k \hat{x}_{k|k-1} \qquad \varepsilon_k = y_k - \hat{y}_k$$

$$K_k = P_{k|k-1} H_k^T S^{-1} \qquad S_k = H_k P_{k|k-1} H_k^T + R_k$$

Optimality Properties

- For a linear model, the Kalman filter provides the WLS solution.
- The KF is the best linear unbiased estimator (BLUE).
- The measurements only affect $\hat{x}$ not P, which can be precomputed.
- It is the Bayes optimal filter for linear model when x_0, v_k, e_k are Gaussian variables,

$$(x_k | y_{k-1}) \sim \mathcal{N}(\hat{x}_{k|k-1}, P_{k|k-1})$$

$$(x_k | y_{1:k}) \sim \mathcal{N}(\hat{x}_{k|k}, P_{k|k})$$

$$\varepsilon_k \sim \mathcal{N}(0, S_k).$$

Robustness and Sensitivity

The following concepts are relevant for all filtering applications, but they are most explicit for Kalman filter:

- **Observability:** Is revealed indirectly by $P_{k|k}$; monitor its rank or better condition number.
- **Divergence tests:** Monitor performance measures and restart the filter after divergence.
- **Outlier rejection:** Monitor sensor observations.
- **Bias error:** Incorrect model gives bias in estimates.
- **Sensitivity analysis:** Uncertain model contributes to the total covariance.
- **Numerical issues:** May give complex estimates.

Observability

1. Snapshot observability if H_k has full rank. WLS can be applied to estimate x.
2. Classical observability for time-invariant and time-varying case,

$$\mathcal{O} = \begin{pmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{pmatrix} \qquad \mathcal{O}_k = \begin{pmatrix} H_{k-n+1} \\ H_{k-n+2}F_{k-n+1} \\ H_{k-n+3}F_{k-n+2}F_{k-n+1} \\ \vdots \\ H_k F_{k-1} \dots F_{k-n+1} \end{pmatrix}.$$

3. The covariance matrix $P_{k|k}$ extends the observability condition by weighting with the measurement noise and to forget old information according to the process noise. Thus, (the condition number of) $P_{k|k}$ is the natural indicator of observability!

Divergence Monitoring

When is $\varepsilon_k \varepsilon_k^T$ significantly larger than its computed expected value $S_k = \mathrm{E}(\varepsilon_k \varepsilon_k^T)$ (note that $\varepsilon_k \sim \mathcal{N}(0, S_k)$)?

Principal reasons

- Model errors
- Sensor model errors: offsets, drifts, incorrect covariances, scaling factor in all covariances
- Sensor errors: outliers, missing data
- Numerical issues

Solutions

- In the first two cases, the filter has to be redesigned.
- In the last two cases, the filter has to be restarted.

Rejecting Outlier

Outlier rejection as a hypothesis test

Let $H_0 : \varepsilon_k \sim \mathcal{N}(0, S_k)$, then

$$T(y_k) = \varepsilon_k^T S_k^{-1} \varepsilon_k \sim \chi^2_{n_{y_k}}$$

if everything works fine, and there is no outlier. If $T(y_k) > h_{P_{\mathrm{fa}}}$, this is an indication of outlier, and the measurement update can be omitted.

In the case of several sensors, each sensor i should be monitored for outliers

$$T(y_k^i) = (\varepsilon_k^i)^T S_k^{-1} \varepsilon_k^i \sim \chi^2_{n_{y_k^i}}.$$

Numerical Issues

Square Root Implementation

Square root implementations implicitly ensure symmetric and positive covariance matrices, and halve the order of the condition number.

Quick fixes

- Impose that the covariance matrix is symmetric

```
P = 0.5*P + 0.5*P'.
```

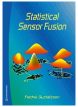
- Use the more numerically stable **Joseph's form** for the measurement update of the covariance matrix:

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^T + K_k R_k K_k^T.$$

- Assure that the covariance matrix is positive definite by setting negative eigenvalues in P to zero or small positive values.
- Avoid singular $R = 0$, even for constraints.
- Increase Q and R if needed (dithering); this can account for all kind of model errors.

Summary

- The Kalman filter is BLUE for linear models and the optimal estimator for linear Gaussian models.
- Consider the following factors when designing your (Kalman) filter:
  - Observability
  - Divergence monitoring
  - Outlier rejection
  - Numerical issues
  - Model uncertainties



Section 7.2–7.7