Kinematic Models

Sensor Fusion

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Purpose

Description of rotational kinematics for sensor fusion applications.

- Rotational kinematics is theoretically a challenging subject.
- Goal to describe the key mathematical background.
- But with a sensor fusion perspective.
- Embed the rotational with translation kinematics to get a complete 3D navigation framework.

Summary of Model Discretization

Linear time-invariant **(LTI) state-space model:**

Continuous time	**Discrete time**
$\dot{x} = Ax + Bu$	$x_{k+1} = Fx_k + Gu_k$
$y = Cx + Du$	$y_k = Hx_k + Ju_k$

u is either input or process noise (then J denotes cross-correlated noise!).

- ***Zero-order hold*** **(ZOH) sampling** assuming the input is piecewise constant:

$$\begin{aligned} x(t+T) &= e^{AT}x(t) + \int_0^T e^{A\tau} Bu(t+T-\tau)\, d\tau \\ &= \underbrace{e^{AT}}_{F} x(t) + \underbrace{\int_0^T e^{A\tau}\, d\tau\, B}_{G} u(t). \end{aligned}$$

- ***First order hold*** **(FOH) sampling** assuming the input is piecewise linear, is another option.

Rotational Kinematics in 3D

Much more complicated in 3D than 2D! Could be a course in itself.
Coordinate notation for rotations of a body in local coordinate system (x, y, z) relative to an earth fixed coordinate system:

Motion components	**Rotation Euler angle**	**Angular speed**
Longitudinal forward motion x	Roll ϕ	ω^x
Lateral motion y	Pitch θ	ω^y
Vertical motion z	Yaw ψ	ω^z

Euler Orientation in 3D

An earth fixed vector $\mathbf{g}$ (for instance the gravitational force) is in the body system oriented as $Q\mathbf{g}$, where

$$
\begin{aligned}
Q &= Q^x_\phi Q^y_\theta Q^z_\psi \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{pmatrix}.
\end{aligned}
$$

Note:

The result depends on the order of rotations $Q^x_\phi Q^y_\theta Q^z_\psi$. Here, the *xyz* rule is used, but there are other options.

Euler Rotation in 3D

When the body rotate with ω, the Euler angles change according to

$$\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} = \begin{pmatrix}\dot{\phi}\\ 0\\ 0\end{pmatrix} + Q_\phi^x \begin{pmatrix}0\\ \dot{\theta}\\ 0\end{pmatrix} + Q_\phi^x Q_\theta^y \begin{pmatrix}0\\ 0\\ \dot{\psi}\end{pmatrix}.$$

The dynamic equation for Euler angles can be derived from this as

$$\begin{pmatrix}\dot{\phi}\\ \dot{\psi}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta)\\ 0 & \cos(\phi) & -\sin(\phi)\\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta)\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}.$$

Singularities when $\theta = \pm\frac{\pi}{2}$, can cause numeric divergence!

Unit Quaternions

- Vector representation: $q = (q^0, q^1, q^2, q^3)^T$.
- Norm constraint of unit quaternion: $\|q\| = q^T q = 1$.
- The quaternion can be interpreted as an axis angle:

$$q = \begin{pmatrix} \cos(\frac{1}{2}\alpha) \\ \sin(\frac{1}{2}\alpha)\hat{v} \end{pmatrix},$$

where q represents a rotation with α around the axis defined by $\hat{v}$, $\|\hat{v}\| = 1$.

Pros and Cons

+ No singularity.
+ No 2π ambiguity.
– More complex and non-intuitive algebra.
– The norm must be maintained; this can be handled by projection or as a virtual measurement with small noise.

Quaternion Orientation in 3D

The orientation of the vector $\mathbf{g}$ in body system is $Q\mathbf{g}$, where

$$Q = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & -2q_0q_1 + 2q_2q_3 \\ -2q_0q_2 + 2q_1q_3 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2q_0^2 + 2q_1^2 - 1 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 + 2q_2^2 - 1 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 2q_0^2 + 2q_3^2 - 1 \end{pmatrix}.$$

Quaternion Rotation in 3D

Rotation with ω gives a dynamic equation for q which can be written in two equivalent forms:

$$\dot{q} = \frac{1}{2} S(\omega) q = \frac{1}{2} \bar{S}(q) \omega,$$

where

$$S(\omega) = \begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{pmatrix}, \qquad \bar{S}(q) = \begin{pmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{pmatrix}.$$

Sampled Form of Quaternion Dynamics

The ZOH sampling formula

$$q(t+T) = e^{\frac{1}{2}S\big(\omega(t)\big)T} q(t)$$

actually has a closed form solution

$$q(t+T) = \Big(\cos(\tfrac{T}{2}\|\omega(t)\|) I_4 + \tfrac{T}{2}\overbrace{\frac{\sin(\frac{T}{2}\|\omega(t)\|)}{\frac{T}{2}\|\omega(t)\|}}^{\operatorname{sinc}(\frac{T}{2}\|\omega(t)\|)} S\big(\omega(t)\big)\Big) q(t)$$
$$\approx \Big(I_4 + \tfrac{T}{2} S\big(\omega(t)\big)\Big) q(t).$$

The approximation coincides with Euler forward sampling approximation, and has to be used in more complex models where, e.g., ω is part of the state vector.

Double Integrated Quaternion

$$\begin{pmatrix} \dot{q}(t) \\ \dot{\omega}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} S(\omega(t)) q(t) \\ w(t) \end{pmatrix}.$$

There is no known closed form discretized model. However, the approximate form can be discretized using the chain rule to

$$\begin{pmatrix} q(t+T) \\ \omega(t+T) \end{pmatrix} \approx \underbrace{\begin{pmatrix} I_4 \frac{T}{2} S\big(\omega(t)\big) & \frac{T}{2} \bar{S}\big(q(t)\big) \\ 0_{3\times 4} & I_3 \end{pmatrix}}_{F(t)} \begin{pmatrix} q(t) \\ \omega(t) \end{pmatrix}$$

$$+ \underbrace{\begin{pmatrix} \frac{T^3}{4} S\big(\omega(t)\big) I_4 \\ T I_3 \end{pmatrix}}_{G(t)} v(t).$$

Rigid Body Kinematics

A "multi-purpose" model for all kind of navigation problems in 3D (22 states)

$$\begin{pmatrix} \dot{p} \\ \dot{v} \\ \dot{a} \\ \dot{q} \\ \dot{\omega} \\ \dot{b}^{\text{acc}} \\ \dot{b}^{\text{gyro}} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}S(\omega) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ v \\ a \\ q \\ \omega \\ b^{\text{acc}} \\ b^{\text{gyro}} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^{a} \\ v^{\omega} \\ v^{\text{acc}} \\ v^{\text{gyro}} \end{pmatrix}.$$

Bias states for the accelerometer and gyroscope have been added as well.

Sensor Model for Kinematic Model

Inertial sensors (gyroscope, accelerometer, magnetometer) are used as sensors.

$$
\begin{aligned}
y_t^{\text{acc}} &= R(q_t)(a_t - \mathbf{g}) + b_t^{\text{acc}} + e_t^{\text{acc}}, & e_t^{\text{acc}} &\sim \mathcal{N}(0, R_t^{\text{acc}}),\\
y_t^{\text{mag}} &= R(q_t)\mathbf{m} + b_t^{\text{mag}} + e_t^{\text{mag}}, & e_t^{\text{mag}} &\sim \mathcal{N}(0, R_t^{\text{mag}}),\\
y_t^{\text{gyro}} &= \omega_t + b_t^{\text{gyro}} + e_t^{\text{gyro}}, & e_t^{\text{gyro}} &\sim \mathcal{N}(0, R_t^{\text{gyro}}).
\end{aligned}
$$

Bias observable, but special calibration routines are recommended:

Stand-still detection: When $\|y_t^{\text{acc}}\| \approx \mathbf{g}$ and/or $\|y_t^{\text{gyro}}\| \approx 0$, the gyro and acc bias is readily read off. Can decrease drift in dead-reckoning from cubic to linear.

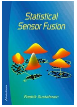
Ellipse fitting: When "waving the sensor" over short time intervals, the gyro can be integrated to give accurate orientation, and acc and magnetometer measurements can be transformed to a sphere from an ellipse.

Summary

- Dynamics for 3D orientation expressed in quaternion q is the most used form in navigation applications $\dot{q} = \frac{1}{2} S(\omega) q = \frac{1}{2} \bar{S}(q) \omega$.
- Discretized approximate model

$$q(t+T) \approx \left(I_4 + \frac{T}{2} S\big(\omega(t)\big) \right) q(t).$$

- Quaternion can be part of a larger model with more states:
    1. Rotational rates ω.
    2. Translational states (p, v, a).
    3. Sensor bias states b.
- Measurements from accelerometers, gyroscopes and magnetometers can then be used as inputs and outputs in a Kalman filter.



Section 13.2 – 13.3